ANNALS OF MATHEMATICAL LOGIC 6 (1973) 147–176. © North-Holland Publishing Company

FINITE AXIOMS OF CHOICE

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Received 2 April 1973

§ 1. Introduction

The finite axioms of choice C_n for n a positive integer, have been extensively studied by Mostowski [5] and Gauntt [1–3], among others. It is on the results and methods of these two authors that the present paper is heavily based. C_n asserts that any set of n-element sets has a choice function, and C_n^0, C_n^* are the modifications of this for ordered and well-ordered sets of n-element sets, respectively. If Z is a finite set of integers, C_Z denotes the conjunction of C_n for $n \in Z$. Similarly for C_Z^0 and C_Z^*. The following two conditions on a finite set Z of integers, and an integer n, were introduced and studied by Mostowski in [5].

$\mathsf{D}(n, Z)$: For any subgroup G of S_n without fixed points, there is a subgroup H of G and proper subgroups $K_1, ..., K_r$ of H such that $\Sigma | H : K_i | \in Z$.

$\mathsf{K}(n, Z)$: For any subgroup G of S_n without fixed points, there is a subgroup H of G^ω (the restricted direct product of countably many copies of G) and proper subgroups $K_1, ..., K_r$ of H such that $\Sigma | H : K_i | \in Z$.

Gauntt also introduced the following condition (in [1]).

$\mathsf{L}(n, Z)$: For any subgroup G of S_n without fixed points, there are proper subgroups $K_1, ..., K_r$ of G such that $\Sigma | G : K_i | \in Z$.

The results known up till now may be summarized as follows:

$\mathsf{D}(n, Z) \Leftrightarrow C_Z \to C_n$ ($\Rightarrow$, Mostowski [5], $\Leftarrow$, Gauntt [2, 3]).

$\mathsf{L}(n, Z) \Leftrightarrow C_Z^* \to C_n^*$ (Gauntt [1].

$\mathsf{K}(n, Z) \Leftarrow C_Z \to C_n^*$ (Mostowski [5], in effect).

We have the following to add to the picture.

$\mathsf{K}(n, Z) \Rightarrow C_Z \to C_n^*$.

$\mathsf{L}(n, Z) \Leftrightarrow C_Z^0 \to C_n^0$ (essentially the same proof is used as Gauntt presumably had for $\mathsf{L}(n, Z) \Leftrightarrow C_Z^* \to C_n^*$).

$C_Z^0 \to C_n^* \Leftrightarrow C_Z^* \to C_n^*$.

$C_Z \to C_n^0 \Leftrightarrow C_Z \to C_n$.

In fact, our results will be generalizations of those shown here, in three ways. Firstly, we shall consider conjunctions of axioms of "different sorts", that is, those of the form $C_{Z_1} \wedge C_{Z_2}^0 \wedge C_{Z_3}^*$ ($\wedge$ denotes "and"). Secondly, we shall allow the Z_i here to be infinite, and give an appropriate definition of the C_{Z_i}, $C_{Z_i}^0$ and $C_{Z_i}^*$ when it is. And thirdly, we shall allow a more general type of finite choice axiom than C_n, C_n^0 and C_n^*. A typical example of this last generalization is "for any set of 7-element sets there is a function choosing a 2-element subset of each".

The main point about our necessary and sufficient conditions is that when the Z_i are finite they are all effectively verifiable. Thus the study of the finite axioms of choice we discuss is reduced to finitary combinatorics.

The suggestion of making the third generalization was made to the author by Prof. A. Levy. We should also like to thank R. Gauntt for sending us a preprint of his paper [3].

We now introduce our notation.

For any set X, $[X]^n$ is the set of n-element subsets of X, and $e(X)$ is the set of non-empty finite subsets of X; thus $e(X) = \bigcup \{[X]^n : 0 < n < \omega\}$, where ω is the set of natural numbers. $S(X)$ is the group of permutations of X.

Let $e_0(X) = X$, $e_{n+1}(X) = e(e_n(X))$ and $e_\omega(X) = \bigcup\{e_n(X) : n \in \omega\}$. The difficulty arises that some elements may appear at more than one level. A formal way round the problem is to attach type superscripts to each member of $e_\omega(X)$. Since this is rather cumbersome, we shall just assume that it could be done in any given case, and that we can distinguish between occurrences of a set at different levels.

Let f map X 1–1 onto Y. Then for each $n \in \omega$, we get the induced map, also denoted by f, from $e_n(X)$ onto $e_n(Y)$, given by $f(\xi) = \{f(\eta) : \eta \in \xi\}$. Clearly, f is 1–1 from $e_0(X)$ to $e_0(Y)$. Assume inductively that it is 1–1 from $e_n(X)$ to $e_n(Y)$, and let $\xi_1, \xi_2 \in e_{n+1}(X)$ satisfy $f(\xi_1) = f(\xi_2)$. Then $\eta_1 \in \xi_1 \to f(\eta_1) \in f(\xi_1) \to f(\eta_1) \in f(\xi_2) \to$ for some $\eta_2 \in \xi_2, f(\eta_1) = f(\eta_2) \to \eta_1 = \eta_2 \in \xi_2$. Therefore $\xi_1 \subseteq \xi_2$ and similarly

$\xi_2 \subseteq \xi_1$. Hence f is 1–1 on $e_{n+1}(X)$, and by induction f is 1–1 on each $e_n(X)$.

A particular case is when $X = Y$ and $f \in S(X)$. We usually use σ, τ for permutations of X (members of $S(X)$). Define $\sim$ on $e_n(X)$ thus. $\xi \sim \eta$ if for some $\sigma \in S(X)$, $\sigma\xi = \eta$. $V_{m,n}$ is the set of $\sim$-classes ($S(n)$-orbits) of $e_m(n)$. $V = \bigcup\{V_{m,n}: m, n \in \omega, n \neq 0\}$. For $v \in V$, let $\langle m(v), n(v)\rangle$ be the (unique) pair $\langle m, n\rangle$ such that $v \in V_{m,n}$.

To show that v uniquely determines $\langle m(v), n(v)\rangle$, one proves by induction on $\min(m_1, m_2)$ that if a non-empty union of members of V_{m_1,n_1} is equal to a non-empty union of members of V_{m_2,n_2}, then $m_1 = m_2$ and $n_1 = n_2$. If $\min(m_1, m_2) = 0$, this follows from the facts that $\{0, 1, ..., n-1\}$ is the only member of $V_{0,n}$, and that $0 \in v \in V_{m,n}$ implies $m = 0$. Otherwise, let $\bigcup A = \bigcup B$, where $\emptyset \neq A \subseteq V_{m_1,n_1}$ and $\emptyset \neq B \subseteq V_{m_2,n_2}$. Then $\{\bigcup x: x \in \bigcup A\}$ is a non-empty union of members of V_{m_1-1,n_1} and also of V_{m_2-1,n_2}. Hence by induction hypothesis, $m_1 - 1 = m_2 - 1$ and $n_1 = n_2$, giving the desired result.
Clearly, if $v \in V_{m,n}$, then $v \in e_{m+1}(n)$.

Suppose that $|X| = n$, and let f map n 1–1 onto X. Then we get the induced map f from $e_{m+1}(n)$ onto $e_{m+1}(X)$ as above, and so $f(v) \in e_{m+1}(X)$, where $v \in V_{m,n}$. It is easily checked that $f(v)$ is a $\sim$-class of $e_m(X)$, and that it is independent of the choice of f. We denote it by $v(X)$.

Examples 1.1. If $v \in V_{0,n}$ and $|X| = n$, then $v(X) = X$. Thus the Mostowski axioms are a special case of those we consider.

If v is the $\sim$-class of n determined by $\{0, 1\}$ and $|X| = n$, then $v(X)$ is the set of 2-element subsets of X.

We are now in a position to formulate the axioms of choice C_Z, C_Z^0, C_Z^* for $Z \subseteq V$.

C_Z: For any set X, $\{v(\xi): v \in Z, \xi \in X, |\xi| = n(v)\}$ has a choice function.

C_Z^0: For any ordered set X, $\{v(\xi): v \in Z, \xi \in X, |\xi| = n(v)\}$ has a choice function.

C_Z^*: For any well-ordered set X, $\{v(\xi): v \in Z, \xi \in X, |\xi| = n(v)\}$ has a choice function.

If $v \in V$, C_v is $C_{\{v\}}$. Similarly C_v^0, C_v^*.

We devote the rest of this section to setting up our necessary and sufficient conditions. Let $\xi \in e_n(X)$. Then $H(\xi) = \{\sigma \in S(X)\colon \sigma\xi = \xi\}$. We adopt some suitable convention for coding ordered pairs of members of $e_\omega(X)$ and choice functions for members of $e_\omega(X)$ as members of $e_\omega(X)$. (An ordered pair of members of $e_\omega(X)$ need not, and a choice function for a member of $e_\omega(X)$ will not, lie in $e_\omega(X)$, not being "stratified". However, it is clear that a suitable copy will lie in $e_\omega(X)$.)

If G is a group of permutations of a set X ($X = n(v)$ in what follows), then G^m is the group which acts on $X \cdot m$ (m copies of X) like G on each coordinate, where m is some positive integer. Let $Z, Z_1, Z_2 \subseteq V$, and $v \in V$. Then $\mathsf{A}(Z, v)$, $\mathsf{B}(Z_1, Z_2, v)$, $\mathsf{C}(Z_1, Z_2, v)$ are the following three conditions.

$\mathsf{A}(Z, v)$: If G is a group of permutations of $n(v)$ which moves every member of v, there is an $X \in e_\omega(n(v))$ such that $|X| = n(w)$ for some $w \in Z$, and for each $\xi \in w(X)$, $H(X) \cap G \not\subseteq H(\xi)$.

$\mathsf{B}(Z_1, Z_2, v)$: If G is a group of permutations of $n(v)$ which moves every member of v, there is an X such that

either $X \in e_\omega(n(v))$, $|X| = n(w)$ for some $w \in Z_1$, and for each $\xi \in w(X)$, $H(X) \cap G \not\subseteq H(\xi)$,

or $X \in e_\omega(n(v) \cdot m)$ for some $m \in \omega$, $|X| = n(w)$ for some $w \in Z_2$, $H(X) = G^m$, and for each $\xi \in w(X)$, $G^m \not\subseteq H(\xi)$.

$\mathsf{C}(Z_1, Z_2, v)$: If G is a group of permutations of $n(v)$ which moves every member of v, then there is an $X \in e_\omega(n(v) \cdot m)$ for some $m \in \omega$, such that

either $|X| = n(w)$ for some $w \in Z_1$, and for each $\xi \in w(X)$, $G^m \cap H(X) \not\subseteq H(\xi)$,

or $|X| = n(w)$ for some $w \in Z_2$, $H(X) = G^m$, and for each $\xi \in w(X)$, $G^m \not\subseteq H(\xi)$.

These conditions are obviously related to $\mathsf{D}(n, Z)$, $\mathsf{K}(n, Z)$ and $\mathsf{L}(n, Z)$, and the precise connection will emerge later. We shall show that

$$\mathrm{C}_{Z_1} \wedge \mathrm{C}^0_{Z_2} \wedge \mathrm{C}^*_{Z_3} \to \mathrm{C}_v \Leftrightarrow \mathsf{A}(Z_1, v),$$
$$\mathrm{C}_{Z_1} \wedge \mathrm{C}^0_{Z_2} \wedge \mathrm{C}^*_{Z_3} \to \mathrm{C}^0_v \Leftrightarrow \mathsf{B}(Z_1, Z_2, v),$$
$$\mathrm{C}_{Z_1} \wedge \mathrm{C}^0_{Z_2} \wedge \mathrm{C}^*_{Z_3} \to \mathrm{C}^*_v \Leftrightarrow \mathsf{C}(Z_1, Z_2 \cup Z_3, v).$$

§ 2. Sufficiency results

Theorem 2.1. $\mathsf{A}(Z, v)$ *is sufficient for* $\mathrm{C}_Z \to C_v$.

Proof. Let X be any set. Examining the definition of C_v, we see that only those members of X with $n(v)$ elements are affected, so we suppose that $\xi \in X \to |\xi| = n(v)$.

For each $\xi \in X$, we define successively $\xi_i \in e_\omega(\xi)$, so that $\xi_0 = \xi$, and for each i

either $H(\xi_i)$ has a fixed point in $v(\xi)$, and $\xi_{i+1} = \xi_i$,

or $H(\xi_i)$ has no fixed point in $v(\xi)$, and $H(\xi_{i+1})$ is a proper subgroup of $H(\xi_i)$.

We shall apply C_Z just a finite number of times, not more than $n(v)!$ in fact.

Suppose then that we have defined ξ_i for each $\xi \in X$. Let

$$X_i = \{\xi \in X\colon H(\xi_i) \text{ has no fixed point in } v(\xi)\}\ .$$

Now if f is a 1–1 map from ξ onto $n(v)$, where $\xi \in X_i$, then $f(\xi_i) \in e_\omega(n(v))$. As $e_\omega(n(v))$ has a natural well-ordering, we may let

$$F^i_\xi = \{f\colon f \text{ is a } 1\!-\!1 \text{ map from } \xi \text{ onto } n(v), \text{ and } f(\xi_i) \text{ is minimal}\}$$

Clearly, if $f \in F^i_\xi$, then $f(H(\xi_i)) = H(f(\xi_i))$ is a group of permutations of $n(v)$ with no fixed points in v, and is independent of the choice of f from F^i_ξ.

By $\mathsf{A}(Z, v)$, there is an $\eta \in e_\omega(n(v))$ such that $|\eta| = n(w)$ for some $w \in Z$, and such that for each $\zeta \in w(\eta)$, $H(\eta) \cap H(f(\xi_i)) \not\subseteq H(\zeta)$. Since $e_\omega(n(v))$ and V are naturally well-ordered, we may choose one such pair, η_ξ, w_ξ, say.

Let $Y_\xi = \{\sigma\eta_\xi\colon \sigma \in S(n(v))\}$, and let $Z_\xi = f^{-1} Y_\xi$, where $f \in F^i_\xi$. Then if $f, f' \in F^i_\xi, f'f^{-1} \in S(n(v))$, so $f^{-1} Y_\xi = (f')^{-1} Y_\xi$, and Z_ξ is independent of the choice of f. Also $\zeta \in Z_\xi \to |\zeta| = n(w_\xi)$.

By C_Z, there is a choice function g for $\{w_\xi(\zeta)\colon \zeta \in Z_\xi, \xi \in X_i\}$. For each $\xi \in X_i$, $g_\xi = g \restriction \{w_\xi(\zeta)\colon \zeta \in Z_\xi\}$ is a choice function for $\{w_\xi(\zeta)\colon \zeta \in Z_\xi\}$.

Let $\xi_{i+1} = \langle \xi_i, g_\xi \rangle$, where ordered pairs and choice functions are coded suitably by some member of $e_\omega(\xi)$, as was mentioned in the introduction.

Then $H(\xi_{i+1})$ is certainly a subgroup of $H(\xi_i)$. To show that it is a proper subgroup it is enough to show that $H(\xi_i) \not\subseteq H(g_\xi)$.

Now for each $\zeta \in w_\xi(\eta_\xi)$, we have $H(\eta_\xi) \cap H(f(\xi_i)) \not\subseteq H(\zeta)$. Thus in particular, $H(\eta_\xi) \cap H(f(\xi_i)) \not\subseteq H(f(g_\xi)[w_\xi(\eta_\xi)])$, where $f(g_\xi)$ is the image of g_ξ under f and is a choice function for $\{w_\xi(f(\zeta)): \zeta \in Z_\xi\}$, and $w_\xi(\eta_\xi)$ is in its domain since $f^{-1}(\eta_\xi) \in Z_\xi$.

Let $\sigma \in H(\eta_\xi) \cap H(f(\xi_i)) - H(f(g_\xi)[w_\xi(\eta_\xi)])$. Then $\sigma \in H(f(\xi_i))$ and $\sigma \notin H(f(g_\xi))$.

Hence $H(f(\xi_i)) \not\subseteq H(f(g_\xi))$, and so $H(\xi_i) \not\subseteq H(g_\xi)$, as desired.

Since the $H(\xi_i)$ are properly decreasing (for fixed ξ) whenever $H(\xi_i)$ has no fixed point in $v(\xi)$, and as $|S(\xi)| = n(v)! = N$, we have now established that $H(\xi_N)$ has a fixed point in $v(\xi)$, for each $\xi \in X$.

As before we may let

$$F_\xi = \{f\colon f \text{ is a } 1-1 \text{ mapping from } \xi \text{ onto } n(v) \\ \text{such that } f(\xi_N) \text{ is minimal}\}\ .$$

Let $g(\xi)$ be the minimal member of v which is the image of a fixed point in $v(\xi)$ of $H(\xi_N)$ under a member of F_ξ, and let $h(v(\xi)) = f^{-1}(g(\xi))$, where $f \in F_\xi$.

Clearly, $f^{-1}(g(\xi)) \in f^{-1}(v) = v(\xi)$, and so we just have to check that $h(v(\xi))$ is independent of the choice of f.

If $f, f' \in F_\xi$, then $f'f^{-1}(f\xi_N) = f'\xi_N = f\xi_N$, so $f'f^{-1} \in H(f\xi_N)$. Therefore $f'f^{-1}$ fixes every fixed point of $H(f\xi_N)$, and $f'f^{-1}g(\xi) = g(\xi)$. Hence $f^{-1}(g(\xi)) = (f')^{-1}(g(\xi))$, as desired.

Before continuing with the next theorem, it is convenient to have the following lemma.

Lemma 2.2. *Let A and B be disjoint finite sets, and $\xi \in e_\omega(A \cup B)$ with $|\xi| = k$. Let $S(B)$, the group of permutations of B, be regarded as a subgroup of $S(A \cup B)$ in the natural way (i.e., as those members of $S(A \cup B)$ which fix A pointwise). Then there is an $\eta \in e_{k+5}(B)$ and a $1-1$ mapping f from ξ onto η such that $H(\xi) \cap S(B) = H(\eta) \cap S(B)$, and for each $x \in \xi$, $H(x) \cap S(B) = H(f(x)) \cap S(B)$.*

Proof. Let **B** be some arrangement of B in a sequence. Then **B** can be regarded as a member of $e_2(B)$. (A sequence can be identified with the set of its non-empty initial segments.) If H is a subgroup of $S(B)$, let $X_H = \{\sigma\mathbf{B} \colon \sigma \in H\}$. It is easily checked that $H(X_H) \cap S(B) = H$.

Consider the $H(\xi) \cap S(B)$-orbits of ξ. Let them be $\xi_0, \ldots, \xi_{n-1}$ and pick $x_i \in \xi_i$ for each i. Notice that $n \leq k$. We now need several copies of a given set, so we let $\{X\}^0 = X$, and $\{X\}^{j+1} = \{\{X\}^j\}$. (This is merely a technical device.)

Let

$$\eta = \{\{\{\{\sigma X_{H(\xi) \cap H(x_i) \cap S(B)}\}^i, \{B\}^{i+2}\}\}^{k-i} \colon \sigma \in H(\xi) \cap S(B), i < n\} .$$

Firstly, $\sigma X_{H(\xi) \cap H(x_i) \cap S(B)} \in e_3(B)$, as $\mathbf{B} \in e_2(B)$, and $\{B\}^2 \in e_3(B)$. Hence

$$\{\{\sigma X_{H(\xi) \cap H(x_i) \cap S(B)}\}^i , \{B\}^{i+2}\} \in e_{i+4}(B) .$$

Therefore

$$\{\{\{\sigma X_{H(\xi) \cap H(x_i) \cap S(B)}\}^i , \{B\}^{i+2}\}\}^{k-i} \in e_{k+4}(B) ,$$

and hence $\eta \in e_{k+5}(B)$ as desired.

By definition of η, $H(\xi) \cap S(B) \subseteq H(\eta) \cap S(B)$. On the other hand, if $\sigma\eta = \eta$, where $\sigma \in S(B)$, then

$$\sigma X_{H(\xi) \cap H(x_0) \cap S(B)} = \tau X_{H(\xi) \cap H(x_0) \cap S(B)}$$

for some $\tau \in H(\xi) \cap S(B)$. (We can recover 0 on the right hand side from the number of brackets.)

Therefore

$$\tau^{-1}\sigma \in H(\xi) \cap H(x_0) \cap S(B) \subseteq H(\xi) \cap S(B) ,$$
$$\sigma \in (\tau \cdot H(\xi) \cap S(B)) = H(\xi) \cap S(B) .$$

Now we map ξ to η by f.

$$f(\sigma x_i) = \{\{\{\sigma X_{H(\xi) \cap H(x_i) \cap S(B)}\}^i, \{B\}^{i+2}\}\}^{k-i} ,$$

where $\sigma \in H(\xi) \cap S(B)$.

To check that f is well-defined, $1-1$, and that $H(f(\sigma x_i)) \cap S(B) = H(\sigma x_i) \cap S(B)$, is straightforward.

Theorem 2.3. $\mathsf{B}(Z_1, Z_2, v)$ *is sufficient for* $\mathrm{C}_{Z_1} \wedge \mathrm{C}^0_{Z_2} \to \mathrm{C}^0_v$.

Proof. The ideas are very similar to those in Theorem 2.1.

We define ξ_i successively as before, with the same properties for each $\xi \in X$, where X is an ordered set of $n(v)$-element sets. That is, $\xi_0 = \xi$, and for each i,

either $H(\xi_i)$ has a fixed point in $v(\xi)$, and $\xi_{i+1} = \xi_i$,

or $H(\xi_i)$ has no fixed point in $v(\xi)$, and $H(\xi_{i+1})$ is a proper subgroup of $H(\xi_i)$.

Once this has been done we may find a choice function for $\{v(\xi)\colon \xi \in X\}$ as in Theorem 2.1.

F^i_ξ is the set of $1-1$ maps f from ξ onto $n(v)$ such that $f(\xi_i)$ is minimal. Make a choice of η_ξ and w_ξ as before. If $\eta_\xi \in e_\omega(n(v))$, then $|\eta_\omega| = n(w_\xi)$, and $w_\xi \in Z_1$, the proof as in Theorem 2.1 applies.

In the other case, $\eta_\xi \in e_\omega(n(v) \cdot m_\xi)$, some $m_\xi \in \omega$, $|\eta_\xi| = n(w_\xi)$ and $w_\xi \in Z_2$. In addition, we have $H(\eta_\xi) = (H(f(\xi_i)))^{m_\xi}$, and for each $\zeta \in w_\xi(\eta_\xi)$, $(H(f(\xi_i)))^{m_\xi} \not\subseteq H(\zeta)$. We let $(F^i_\xi)^{m_\xi}$ be the set of all $1-1$ maps f from $\xi \cdot m_\xi$ onto $n(v) \cdot m_\xi$ which send the j^{th} copy of ξ $1-1$ onto the j^{th} copy of $n(v)$ by f_j, say, for each $j < m_\xi$, and such that each f_j is in F^i_ξ.

Let $Z_\xi = f^{-1}\eta_\xi$, where $f \in (F^i_\xi)^{m_\xi}$. Z_ξ is independent of the choice of f from $(F^i_\xi)^{m_\xi}$, for if $f, f' \in (F^i_\xi)^{m_\xi}$, then $f_j(\xi_i) = f'_j(\xi_i)$ for each $j < m_\xi$, and $f'_j f_j^{-1} \in H(f_j(\xi_i))$. Therefore $f' f^{-1} \in (H(f_j(\xi_i)))^{m_\xi} = H(\eta_\xi)$, and $f^{-1}\eta_\xi = (f')^{-1}\eta_\xi$.

Let X_i be the set of all members of X for which this case arises. Then $\{Z_\xi\colon \xi \in X_i\}$ is an ordered set, and for each $\xi \in X_i$, $|Z_\xi| = n(w_\xi)$ and $w_\xi \in Z_2$. By $\mathrm{C}^0_{Z_2}$, there is a choice function g for $\{w_\xi(Z_\xi)\colon \xi \in X_i\}$.

Thus for each $\xi \in X_i$, $g(w_\xi(Z_\xi)) \in w_\xi(Z_\xi)$, and so for any $f \in (F^i_\xi)^{m_\xi}$, $f(g(w_\xi(Z_\xi))) \in w_\xi(\eta_\xi)$. Choose the least member t_ξ of $w_\xi(\eta_\xi)$ such that for some $f \in (F^i_\xi)^{m_\xi}$, $f(g(w_\xi(Z_\xi))) = t_\xi$, and let $\bar{F}^i_\xi$ be the set of members f of $(F^i_\xi)^{m_\xi}$ such that $f(g(w_\xi(Z_\xi))) = t_\xi$.

Since $t_\xi \in w_\xi(\eta_\xi)$, $(H(f(\xi_i)))^{m_\xi} \not\subseteq H(t_\xi)$. Therefore there is a least coordinate j such that

$$[H(f(\xi_i))]_j \not\subseteq H(t_\xi)\,.$$

By Lemma 2.2, there is a $\zeta \in [e_\omega(n(v))]_j$ such that

$$[H(f(\xi_i))]_j \cap H(\zeta) = [H(f(\xi_i))]_j \cap H(t_\xi) .$$

Let ζ_ξ be the least such ζ, and hence

$$[H(f(\xi_i))]_j \not\subseteq H(\zeta_\xi) .$$

Let $\xi_{i+1} = \langle \xi_i, f_j^{-1}\zeta_\xi\rangle$ for $f \in \bar{F}^i_\xi$. (f_j denotes the j^{th} component of f.)

As $H(\zeta_\xi) \cap [H(f(\xi_i))]_j$ is a proper subgroup of $[H(f(\xi_i))]_j$, $H(\langle \xi_i, f_j^{-1}\zeta_\xi\rangle)$ is a proper subgroup of $H(\xi_i)$. So we must just check that ξ_{i+1} is independent of the choice of f from $\bar{F}^i_\xi$.

If $f, f' \in \bar{F}^i_\xi$, then

$$f'f^{-1}(t_\xi) = f'f^{-1}(f(g(w_\xi(Z_\xi)))) = f'(g(w_\xi(Z_\xi))) = t_\xi .$$

Hence $(f'f^{-1})_j \in [H(f(\xi_i))]_j \cap H(\zeta_\xi)$, and so $f_j^{-1}\zeta_\xi = (f')_j^{-1}\zeta_\xi$.

Theorem 2.4. $\mathrm{C}(Z_1, Z_2, v)$ *is sufficient for* $\mathrm{C}_{Z_1} \wedge \mathrm{C}^*_{Z_2} \to \mathrm{C}^*_v$.

Proof. Again the proof follows the same sort of pattern.

Let $\langle X, <\rangle$ be a well-ordered set of $n(v)$-element sets. We may suppose that any two members of X are disjoint.

ξ_i is defined as before. Suppose then that ξ_i, F^i_ξ, η_ξ and m_ξ have all been chosen for a particular i.

If $\eta_\xi \in e_\omega(n(v)\cdot m_\xi)$, $|\eta_\xi| = n(w_\xi)$ and $w_\xi \in Z_2$, we argue just as in the proof of Theorem 2.3, except that $\{Z_\xi : \xi \in X_t\}$ is now well-ordered rather than ordered.

In the other case, $\eta_\xi \in e_\omega(n(v)\cdot m_\xi)$, $|\eta_\xi| = n(w_\xi)$, $w_\xi \in Z_1$ and for each $\zeta \in w_\xi(\eta_\xi)$,

$$(H(f(\xi_i)))^{m_\xi} \cap H(\eta_\xi) \not\subseteq H(\zeta) .$$

For each $\eta \in e_\omega(n(v)\cdot m)$ (some $m \in \omega$), $w \in Z_1$ and $H \subseteq S(n(v))$, let

$$X(\eta, w, H) = \{\xi \in X \colon \eta_\xi = \eta, w_\xi = w, H(f(\xi_i)) = H, \text{ each } f \in F^i_\xi\} .$$

It is crucial to the argument that we should only need to consider finitely many non-empty $X(\eta, w, H)$. To see this we observe firstly that we need only consider finitely many values of m and w since $S(n(v))$ has only finitely many subgroups. To show that for a given m, w and H, only finitely many η can arise we use Lemma 2.2. η satisfies the following. $\eta \in e_\omega(n(v)\cdot m)$, $|\eta| = n(w)$ and for each $\zeta \in w(\eta)$, $H^m \cap H(\eta) \not\subseteq H(\zeta)$.

By Lemma 2.2, there is an $\eta' \in e_{n(w)+5}(n(v)\cdot m)$ and a 1–1 mapping f from η onto η' such that $H(\eta) = H(\eta')$ (as $A = \emptyset$ in this case) and for every $x \in \eta$, $H(x) = H(f(x))$. An easy induction shows that in fact for each $x \in e_\omega(\eta)$, $H(x) = H(f(x))$, and in particular this holds for each $x \in w(\eta)$. In other words, there is an η' in a fixed finite set, $e_{n(w)+5}(n(v)\cdot m)$, having the desired properties too. Since every η we chose is minimal, there are only finitely many of them.

Now let η, w, H be fixed, and let X_0 be an m-element subset of $X(\eta, w, H)$. Let $F^i(X_0)$ be the set of all 1–1 maps f from $\bigcup X_0$ onto $n(v)\cdot m$ whose restriction f_j to the j^{th} member ξ_j of X_0 (in the well-ordering of X) maps ξ_j onto the j^{th} copy of $n(v)$ and lies in $F^i_{\xi_j}$. (This is why we specified that any two members of X should be disjoint.) Then if $f \in F^i(X_0)$, $f^{-1}\eta \in e_\omega(\bigcup X_0)$. Let $Z(X_0) = \{f^{-1}\eta : f \in F^i(X_0)\}$. Then each member of $Z(X_0)$ has $|\eta|$ members.

Using C_{Z_1}, we know that there is a choice function f for

$$\{w(\zeta) \colon \zeta \in Z(X_0),\ X_0 \in [X(\eta, w, H)]^m,\ \langle \eta, w, H\rangle \in T\},$$

where T is the appropriate finite set of triples $\langle \eta, w, H\rangle$ we chose above. (We do not actually need the fact that T is finite at this point.)

Then, for each X_0, $g_{X_0} = g \restriction \{w(\zeta) \colon \zeta \in Z(X_0)\}$ is a choice function for $\{w(\zeta) \colon \zeta \in Z(X_0)\}$.

Now let $f \in F^i(X_0)$, and ξ be the j^{th} member of X_0. Then $H(f_j(\xi_i)) = H$, and so

$$(H(f_j(\xi_i)))^m \cap H(\eta) \not\subseteq H(f(g_{X_0}(w(f^{-1}(\eta))))) .$$

since $f(g_{X_0}(w(f^{-1}(\eta)))) \in w(\eta)$.

Let

$$\sigma \in (H(f_j(\xi_i)))^m \cap H(\eta) - H(f(g_{X_0}(w(f^{-1}(\eta))))) .$$

Then $f^{-1}(\sigma)$ lies in $\Pi\{H(\xi_i)\colon \xi \in X_0\}$ and in $H(f^{-1}\eta)$. But if $f^{-1}(\sigma) \in H(g_{X_0})$, then $\sigma \in H(f(g_{X_0}))$, contrary to the choice of σ.

Therefore $\Pi\{H(\xi_i)\colon \xi \in X_0\} \not\subseteq H(g_{X_0})$. Hence for some $\xi \in X_0$, $\xi = \xi(X_0)$, say (taking the least in the well-ordering of X) $H(\xi_i) \not\subseteq H(g_{X_0})$.

Let $\overline{F}^i(X_0)$ be the set of all members of $F^i(X_0)$ which map g_{X_0} to a minimal member of $e_\omega(n(v)\cdot m)$, among all possible images. Then if $f \in \overline{F}^i(X_0)$, and $\xi(X_0)$ is the j^{th} member of X_0, $H(f_j(\xi_i)) \not\subseteq H(f(g_{X_0}))$

By Lemma 2.2, there is a $\zeta \in e_\omega(n(v))$ such that

$$[H(f(g_{X_0}))]_j = H(f(g_{X_0})) \cap S(n(v)\cdot\{j\}) = H(\zeta)\,.$$

Let $\zeta(X_0)$ be the minimal such ζ in $e_\omega(n(v))$, and let $t(X_0) = \langle f_j^{-1}(\zeta(X_0)), \xi_i\rangle$, where $f \in \overline{F}^i(X_0)$. As before $t(X_0)$ is independent of the choice of f, and $t(X_0) \in e_\omega(\xi(X_0))$. Also,

$$H(t(X_0)) = H(\xi_i) \cap H(f^{-1}(\zeta(X_0))) = H(g_{X_0}) \cap H(\xi_i)\,,$$

and so $H(t(X_0))$ is a proper subgroup of $H(\xi_i)$.

Now let $X'(\eta, w, H)$ be the set of those members ξ of $X(\eta, w, H)$ such that for some $X_0 \in [X(\eta, w, H)]^m$, $\xi = \xi(X_0)$. Then it is clear that $X(\eta, w, H)$ cannot have more than $m-1$ members which do not lie in $X'(\eta, w, H)$. As T is finite, there will only be finitely many such for varying η, w, H. For these we choose ξ_{i+1} arbitrarily, so that $H(\xi_{i+1})$ is a proper subgroup of $H(\xi_i)$. This is why we needed T to be finite.

If $\xi \in X'(\eta, w, H)$, let X_0 be the least member of $[X(\eta, w, H)]^m$ in the lexicographic well-ordering of it such that $\xi = \xi(X_0)$. Then let $\xi_{i+1} = t(X_0)$.

The proof is now complete as in Theorems 2.1 and 2.3.

§3. Necessity results

For Theorems 3.2 and 3.9, which are concerned with necessity conditions connected with $\mathsf{A}(Z, v)$ and $\mathsf{B}(Z_1, Z_2, v)$ respectively, we use constructions very much like that of Gauntt in [3]. Theorem 3.2 is the more directly like his, and for Theorem 3.9, we use the same idea (of adding a generic choice function), but in addition ensuring that a certain collection of subsets of U, the urelemente, can be ordered.

"Urelemente", or "atoms", are sets which have no members, but which are distinct from the empty set. Their existence thus violates the axiom of extensionality (and we let FM be the modification of ZF in which the axiom of extensionality is altered to allow the existence of urelemente). FMC is FM together with the axiom of choice. (ZF, of course, is Zermelo–Fraenkel set theory without the axiom of choice; FM stands for Fraenkel–Mostowski.)

Urelemente are very useful for independence proofs about the axiom of choice because they are set-theoretically "indistinguishable", and hence in particular there can be no definable way of well-ordering them. Pincus showed in [6] that for a wide class of statements φ, a proof of the independence of φ from FM can be converted into a proof of the independence of φ from ZF. The statements we are concerned with in fact come into this category. If, however, one wishes to prove the results directly, one can replace U, the set of urelemente, by a set of sets of generic reals, as remarked by Gauntt in [3].

In view of this discussion, we feel justified in using urelemente freely, as they considerably simplify matters. Let $\mathfrak{M}$ be a countable transitive model of FMC containing an infinite set U of urelemente. The following lemma is the heart of Theorems 3.2 and 3.9, and corresponds to Gauntt's Lemma 5 in [3].

Lemma 3.1. *Let A and B be disjoint finite subsets of U, $Z \subseteq V$ and $v \in V$ be such that $|B| = n(v)$. Suppose that G is a group of permutations of B which moves every member of $v(B)$ and such that for any $X \in e_\omega(B)$ with $|X| = n(w)$, where $w \in Z$, there is a $\xi \in w(X)$ such that $H(X) \cap G \subseteq H(\xi)$. For any set Y, let $T_Z(Y)$ be the set of $n(w)$-element subsets of the set of all pairs of the form $\langle \eta, i \rangle$, where η is a finite set of $1-1$ sequences of members of Y and $i \in \omega$ for $w \in Z$. Then any choice function f for*

$\{w(\xi): \xi \in T_Z(A), |\xi| = n(w)\}$ can be extended to a choice function g for $\{w(\xi): \xi \in T_Z(A \cup B), |\xi| = n(w)\}$ such that $G \subseteq H(g)$, where H is defined with respect to $S(B)$, the group of permutations of B (which fixes A and ω pointwise).

Proof. We are given that if $X \in e_\omega(B)$, where $|X| = n(w)$ for some $w \in Z$, then there is a $\xi \in w(X)$ such that $H(X) \cap G \subseteq H(\xi)$. By Lemma 2.2, the same holds for any member X of $e_\omega(A \cup B \cup \omega)$ since H is defined with respect to $S(B)$. For each X, let $\xi = \xi(X)$ be the least such in some fixed well-ordering of $e_\omega(A \cup B \cup \omega)$.

Let D be the set of G-orbits of $T_Z(A \cup B) - T_Z(A)$, and for each $d \in D$ let $h(d)$ be the least member of d.

If $X \in T_Z(A \cup B)$, $|X| = n(w)$ and $w \in Z$, then we define $g(w(X))$ in the following way. If $X \in T_Z(A)$, then $g(w(X)) = f(w(X))$. Otherwise, $X = \sigma h(d)$ for some (unique) $d \in D$ and some (not necessarily unique) $\sigma \in G$. Let $g(w(X)) = \sigma\xi(h(d))$.

g is well-defined since if $\sigma h(d) = \tau h(d)$, where $\sigma, \tau \in G$, then $\tau^{-1}\sigma \in G \cap H(h(d)) \subseteq H(\xi(h(d)))$.

Secondly, g is a choice function for $\{w(\xi): \xi \in T_Z(A \cup B), |\xi| = n(w)\}$ since $g(w(X)) \in \sigma w(h(d)) = w(\sigma h(d)) = w(X)$.

Finally, $G \subseteq H(g)$ by definition of g.

Theorem 3.2. $\mathsf{A}(Z, v)$ *is necessary for the implication* $\mathrm{C}_Z \wedge \mathrm{C}^0_{Z'} \to \mathrm{C}_v$.

Proof. We add to $\mathfrak{M}$ a generic choice function F as follows. A condition p is a choice function for a subset of $\{w(\xi): \xi \in T_Z(U), w \in Z, |\xi| = n(w)\}$ which involves only finitely many members of U. For the definition of $T_Z(U)$ see the statement of Lemma 3.1. $\mathbf{P}$ is the set of all conditions, partially ordered by inclusion. $\mathfrak{J}$ is an $\mathfrak{M}$-generic filter on $\mathbf{P}$, whose existence is guaranteed by the countability of $\mathfrak{M}$, and $F = \bigcup\mathfrak{J}$ is a generic choice function for $\{w(\xi): \xi \in T_Z(U), w \in Z, |\xi| = n(w)\}$.

For this approach to forcing the reader is referred to Shoenfield's paper [8].

$\mathfrak{N}$ is the submodel of $\mathfrak{M}[\mathfrak{J}]$ consisting of all its members which are hereditarily ordinal definable over $U \cup \{F\}$.

That is, $\xi \in \mathfrak{N} \leftrightarrow \xi \subseteq \mathfrak{N}$ and $\xi \in \mathfrak{M}[\mathfrak{J}]$, and for some formula φ involving ordinals and F as parameters, and for some $u_1, u_2, ..., u_n \in U$,

$$\xi = \{\eta: \varphi(\eta, u_1, ..., u_n)\}, \text{(or } \xi \in U) .$$

Now the class Φ of formulae with ordinals and F as parameters can be well-ordered inside $\mathfrak{N}$. We fix a well-ordering for it. Let $\Vdash$ denote the weak forcing relation, and if p is a condition and $A \subseteq U$, then p_A is the part of p involving only members of A. The following lemma, Lemma 3.3, is standard, and Lemma 3.4 is an adaptation of [3, Lemma 4].

Lemma 3.3. *Suppose that $\varphi \in \Phi$, and $\mathsf{p} \Vdash \varphi(u_1, ..., u_n)$. Then $\mathsf{p}_A \Vdash \varphi(u_1, ..., u_n)$, where $A = \{u_1, ..., u_n\}$.*

Lemma 3.4. *Let $X \in \mathfrak{N}$, and suppose that X is $\{F\}$-ordinal definable over each of A and B, where A and B are finite subsets of U. Then X is $\{F\}$-ordinal definable over $A \cap B$.*

Proof. Suppose that $A \nsubseteq B$ and let $u \in A - B$. We show that X is $\{F\}$-ordinal definable over $A - \{u\}$. A simple induction (treating similarly the case when $B \nsubseteq A$) gives the result as stated.

Let

$$X = \{x: \varphi(x, u, u_1, ..., u_n)\} = \{x: \psi(x, u_1', ..., u_m')\},$$

where $\varphi, \psi \in \Phi$, and $A = \{u, u_1, ..., u_n\}$, $B = \{u_1', ..., u_m'\}$.

Let p in $\mathfrak{P}$ force

$$\forall x(x \in X \leftrightarrow \varphi(x, u, u_1, ..., u_n) \leftrightarrow \psi(x, u_1', ..., u_m')). \tag{1}$$

Then the statement $\mathsf{p}_A \subseteq F$ holds in $\mathfrak{N}$, and as it only involves members of A it may be written in the form $\theta(u, u_1, ..., u_n)$, where $\theta \in \Phi$.

Let $\chi(x, x_1, ..., x_n)$ be the following formula:

$$(\exists y \in U)\,(y \neq x_1 \wedge ... \wedge y \neq x_n \wedge \theta(y, x_1, ..., x_n) \\ \wedge \varphi(x, y, x_1, ..., x_n)).$$

We shall show that $X = \{x: \chi(x, u_1, ..., u_n)\}$, and thus that X is $\{F\}$-ordinal definable over $A - \{u\}$. This is all that is required.

Firstly, if $x \in X$, there is a y, namely u, such that

$$y \neq u_1 \wedge ... \wedge y \neq u_n \wedge \theta(y, u_1, ..., u_n) \wedge \varphi(x, y, u_1, ..., u_n).$$

Therefore $X \subseteq \{x: \chi(x, u_1, ..., u_n)\}$.

Conversely, we suppose that $\chi(x, u_1, ..., u_n)$, where $x \notin X$, and derive a contradiction.

This means that $\chi(x, u_1, ..., u_n)$ and $\neg\varphi(x, u, u_1, ..., u_n)$. By definition of χ, there is a $y \in U$ not equal to any u_i such that $\theta(y, u_1, ..., u_n)$ and $\varphi(x, y, u_1, ..., u_n)$.

Clearly, $y \neq u$ since $\varphi(x, y, u_1, ..., u_n)$ and $\neg\varphi(x, u, u_1, ..., u_n)$, and we also have $\neg(\forall x)(\varphi(x, y, u_1, ..., u_n) \leftrightarrow \varphi(x, u, u_1, ..., u_n))$. As this holds in $\mathfrak{N}$ it is forced by some $\mathsf{q} \in \mathfrak{J}$. By Lemma 3.3,

$$(2) \qquad \mathsf{q}_{A \cup \{y\}} \Vdash \neg(\forall x)(\varphi(x, y, u_1, ..., u_n) \leftrightarrow \varphi(x, u, u_1, ..., u_n)) .$$

If $y \in B$, let $y' \in U-(A \cup B)$, and let σ be the permutation of U which interchanges y and y' and fixes everything else.

If $y \notin B$, let σ be the identity.

By (1), $\mathsf{p}_{A \cup B} \Vdash (\forall x)(\varphi(x, u, u_1, ..., u_n) \leftrightarrow \psi(x, u_1', ..., u_m'))$. By the usual permutational arguments (for details see [8]),

$$(3) \qquad \sigma\mathsf{p}_{A \cup B} \Vdash (\forall x)(\varphi(x, u, u_1, ..., u_n) \leftrightarrow \psi(x, \sigma u_1', ..., \sigma u_m')) .$$

Then neither y nor u is equal to any $\sigma u_i'$, and so if τ is the permutation of U which just interchanges y and u, we have

$$(4) \qquad \tau\sigma\mathsf{p}_{A \cup B} \Vdash (\forall x)(\varphi(x, y, u_1, ..., u_n) \leftrightarrow \psi(x, \sigma u_1', ..., \sigma u_m')) .$$

Suppose we show that $\mathsf{q}_{A \cup \{y\}} \cup \sigma\mathsf{p}_{A \cup B} \cup \tau\sigma\mathsf{p}_{A \cup B}$ is a condition. Then it forces (2), (3) and (4), giving the required contradiction.

It is therefore enough to show that $\mathsf{q}_{A \cup \{y\}}$, $\sigma\mathsf{p}_{A \cup B}$, and $\tau\sigma\mathsf{p}_{A \cup B}$ are pairwise compatible.

Now $\sigma\mathsf{p}_{A \cup B}$ does not involve y, and $\tau\sigma\mathsf{p}_{A \cup B}$ does not involve u. Hence if $Y \in$ domain $\sigma\mathsf{p}_{A \cup B} \cap$ domain $\tau\sigma\mathsf{p}_{A \cup B}$, then Y involves neither y nor u, so is fixed by τ. Thus $\sigma\mathsf{p}_{A \cup B}$ and $\tau\sigma\mathsf{p}_{A \cup B}$ are compatible.

Secondly, let $Y \in$ domain $\mathsf{q}_{A \cup \{y\}} \cap$ domain $\sigma\mathsf{p}_{A \cup B}$. y is not involved in $\sigma\mathsf{p}_{A \cup B}$, and so $Y \in$ domain q_A. But σ fixes A pointwise, and so fixes p_A. Therefore $\mathsf{q}_A(Y) = F(Y) = (\sigma\mathsf{p}_{A \cup B})(Y)$. Thus $\mathsf{q}_{A \cup \{y\}}$ and $\sigma\mathsf{p}_{A \cup B}$ are compatible.

Finally, let $Y \in$ domain $\mathsf{q}_{A \cup \{y\}} \cap$ domain $\tau\sigma\mathsf{p}_{A \cup B}$. As u is not involved in $\tau\sigma\mathsf{p}_{A \cup B}$, $Y \in$ domain $\mathsf{q}_{A'} \cap$ domain $\tau\sigma\mathsf{p}_A$, where $A' = (A - \{u\}) \cup \{y\}$.

Now $\theta(y, u_1, ..., u_n)$ holds, and so $\tau\sigma \mathrm{p}_A \subseteq F$ (since $\tau\sigma\mathrm{p}_A$ is the result of substituting y for u in p_A). Since F is a function, this gives $\mathrm{q}_{A'}(Y) = (\tau\sigma\mathrm{p}_A)(Y)$. Hence $\mathrm{q}_{A\cup\{y\}}$ and $\tau\sigma\mathrm{p}_{A\cup B}$ are compatible.
This completes the proof of the lemma.

By Lemma 3.4, for each $X \in \mathfrak{N}$, there is a unique minimal finite $A \subseteq U$ such that X is $\{F\}$-ordinal definable over A. Let this A be $s(X)$ ("support" of X), and let φ_X be the least formula φ in Φ which defines X from parameters in A. $s'(X)$ is the set of all arrangements (i.e. orderings) $\langle u_1, ..., u_n\rangle$ of A such that $X = \{x\colon \varphi_X(x, u_1, ..., u_n)\}$. One checks in the usual way that the mappings s, s' and the map which takes X to φ_X are all in $\mathfrak{N}$ (i.e., are definable classes). Let this last map be denoted by Ω.

Now let X be any set in $\mathfrak{N}$. We show that $\{w(\xi)\colon w \in Z, \xi \in X, |\xi| = n(w)\}$ has a choice function in $\mathfrak{N}$.

For $\xi \in X$, let $f_1(\xi) = \{\langle A, i\rangle\colon$ for some $\eta \in \xi, s'(\eta) = A$, and φ_η is the i^{th} formula φ in Φ such that for some $\eta_1 \in \xi, \varphi = \varphi_{\eta_1}$ and $s'(\eta_1) = A\}$.

For $\eta \in \xi \in X$, let $f_2(\eta) = \langle s'(\eta), i\rangle$, where φ_η is the i^{th} formula φ in Φ such that for some $\eta_1 \in \xi, \varphi = \varphi_{\eta_1}$ and $s'(\eta_1) = s'(\eta)$.

Then $f_1(\xi) = f_2''\xi$, and this has $|\xi|$ members. Also $s'(\eta)$ is a finite set of $1-1$ sequences of members of U, and so if $|\xi| = n(w)$, where $w \in Z$, then $f_1(\xi) \in T_Z(U)$.

We therefore let $g(w(\xi)) = f_2^{-1}(F(w(f_1(\xi))))$ whenever $|\xi| = n(w)$ and $w \in Z$. Then g is a choice function for $\{w(\xi)\colon w \in Z, \xi \in X, |\xi| = n(w)\}$. Also $g \in \mathfrak{N}$ since it is defined in terms of s', Ω and $\langle \Phi, <\rangle$.

This shows that C_Z holds in $\mathfrak{N}$. Now we show that $\mathrm{C}^0_{Z'}$ holds for any Z'. We remark that this is implied by the following much stronger statement: Any ordered union of orderable sets can be well-ordered (since any finite set is orderable). In ZF this statement implies AC, as is shown by Rubin and Rubin in [7, p. 77], and so we should replace it by "any ordered union of orderable sets can be ordered", which is still clearly strong enough to prove $\mathrm{C}^0_{Z'}$.

Lemma 3.5. *Any ordered partition of U in $\mathfrak{N}$ is finite.*

We omit the proof of this. See [9, Theorem 5(iv)] where the same thing is proved in a very similar situation. ($|U| \in \Delta_2$ in the notation used there.) In fact, the model used here is a submodel of the one discussed there.

Lemma 3.6. (i) *If X is a set such that any ordered partition of X is finite, then any ordered subset of $e(X)$ is finite.*

(ii) *If any ordered subset of $e(X)$ is finite, then any ordered subset of $e_2(X)$ is finite.*

Proof. (i) This is proved in [9, Theorem 4].

(ii) Let Y be an ordered subset of $e_2(X)$, and let $Y' = \{\bigcup\eta\colon \eta \in Y\}$. For each $\eta' \in Y'$, if $\eta \in Y$ satisfies $\bigcup\eta = \eta'$, then $\eta \in P(\eta')$, so there are only finitely many such. As Y is ordered, an ordering of Y' is thereby induced. As $Y' \subseteq e_1(X) = e(X)$, Y' is finite. Hence also Y is finite.

Lemma 3.7. *If the set X in $\mathfrak{N}$ can be ordered, then it can be well-ordered.*

Proof. Using the functions s' and Ω, we may regard X as a set of pairs $\langle \xi, \alpha\rangle$, where ξ is a finite set of $1-1$ sequences of members of U, and α is an ordinal. A sequence of members of U can be regarded as a member of $e_2(U)$ (when coded by the set of its non-empty initial segments), and so $\xi \in e_3(U)$. Thus $X \subseteq e_3(U)$.

Let Y be the set of first coordinates of members of X. Then as X can be ordered, so can Y. Since $Y \subseteq e_3(U)$, Lemma 3.5 and 3.6 show that Y is finite. Therefore X can be well-ordered.

Lemma 3.8. *In $\mathfrak{N}$, a well-ordered union of well-orderable sets can be well-ordered.*

Proof. Let X be well-orderable in $\mathfrak{N}$. Then for some finite $A \subseteq U$, $\langle X, <\rangle$ is $\{F\}$-ordinal definable over A, where $<$ well-orders X. If $\xi \in X$, ξ is thus $\{F\}$-ordinal definable over A too. Hence $s(\xi) \subseteq A$. Therefore $\bigcup\{s(\xi)\colon \xi \in X\}$ is finite. Now let X be a well-ordered set of well-orderable sets. For each $\xi \in X$, $t(\xi) = \bigcup\{s(\eta)\colon \eta \in \xi\}$ is finite, and $\{t(\xi)\colon \xi \in X\}$ can be well-ordered. By Lemmas 3.5 and 3.6, its union is finite. Hence $\bigcup\{s(\eta)\colon \eta \in \bigcup X\}$ is finite, and so $\bigcup X$ can be well-ordered in $\mathfrak{N}$.

Lemmas 3.7 and 3.8 clearly show that any ordered union of orderable sets can be well-ordered, as desired.

To complete the proof of Theorem 3.2, we show that if $\mathsf{A}(Z, v)$ is false, then C_v fails in $\mathfrak{N}$. In fact, we show that if U_1 is an infinite subset of U in $\mathfrak{N}$, then $\{v(\xi)\colon \xi \in [U_1]^{n(v)}\}$ has no choice function in $\mathfrak{N}$.

For suppose that f is such a choice function, and let A be the set of members of U occurring in p, $s(f)$ and $s(U_1)$, where p is a condition forcing "f is a choice function for $\{v(\xi)\colon \xi \in [U_1]^{n(v)}\}$". Let $B \subseteq U_1 - A$, and $|B| = n(v)$, using U_1 infinite and A finite. Let q_1 be a choice function for $\{w(\xi)\colon \xi \in T_Z(A),\ |\xi| = n(w)\}$ extending p.

Since $\mathsf{A}(Z, v)$ fails, there is a group G of permutations of B which moves every member of $v(B)$, and such that for any $X \in e_\omega(B)$ with $|X| = n(w)$, where $w \in Z$, there is a $\xi \in w(X)$ such that $H(X) \cap G \subseteq H(\xi)$.

By Lemma 3.1, there is an extension q of q_1 to $\{w(\xi)\colon \xi \in T_Z(A \cup B),\ |\xi| = n(w)\}$ such that $G \subseteq H(q)$. Hence for each $x \in v(B)$, there is a permutation of $A \cup B$ fixing $A \cup \omega$ pointwise, fixing q, and moving x.

As $q \geq \mathsf{p}$, $q \Vdash$ "f is a choice function for $\{v(\xi)\colon \xi \in [U_1]^{n(v)}\}$". As $v(B) \in$ domain f, we must have $r \Vdash f(v(B)) = x$ for some $x \in v(B)$ and $r \geq q$. Let C be the set of all members of $U - (A \cup B)$ involved in r. By choice of q, there is a permutation σ of $A \cup B$ fixing A pointwise, fixing q, and moving x. Extend σ to U, so that $\sigma''C$ is disjoint from C. Then clearly r and σr are compatible.

Therefore $r \Vdash f(v(B)) = x$ and $\sigma r \Vdash f(v(B)) = \sigma x$. This, together with $r \geq q$, $q \Vdash$ "f is a function", and r, σr compatible, gives a contradiction, and proves the theorem. Notice that it is only at this last point that we need infinite conditions. Everywhere else finite ones would do.

Theorem 3.9. *$\mathsf{B}(Z_1, Z_2, v)$ is necessary for the implication* $\mathrm{C}_{Z_1} \wedge \mathrm{C}^0_{Z_2} \wedge \mathrm{C}^*_{Z_3} \to \mathrm{C}^0_v$.

Proof. The proof will parallel the proof of Theorem 3.2 in many ways, but at several stages will be rather more complicated. Suppose then that $\mathsf{B}(Z_1, Z_2, v)$ fails.

We let $\mathfrak{M}$ and U be as before, except that now we suppose in addition that U is countable in $\mathfrak{M}$. Thus if $\mathbf{Q}$ is the set of rational numbers, we may index U by $\mathbf{Q} \times N$, where $N = n(v)$. Thus we let $U = \{u_{qi}\colon q \in \mathbf{Q}, i \in N\}$.

Let $U_q = \{u_{qi}\colon i \in N\}$ for each q, and let $U' = \{U_q : q \in \mathbf{Q}\}$. U' receives the ordering of $\mathbf{Q}$. We let $\mathbf{U}_q = \langle u_{q0}, \ldots, u_{qN-1}\rangle$. Since $\mathsf{B}(Z_1, Z_2, v)$

fails, there is a group G of permutations of N which moves every member of υ, and such that for every $X \in e_\omega(N)$, if $|X| = n(w)$, where $w \in Z_1$, then there is a $\xi \in w(X)$ such that $H(X) \cap G \subseteq H(\xi)$, and for every $X \in e_\omega(N \cdot m)$, any $m \in \omega$, if $|X| = n(w)$, where $w \in Z_2$, and $H(X) = G^m$, then there is a $\xi \in w(X)$ such that $G^m \subseteq H(\xi)$.

If $\sigma \in G$ and $q \in \mathbf{Q}$, then σ_q is the permutation of U leaving every U_r pointwise fixed for $r \neq q$, and given by $\sigma_q u_{qi} = u_{q(\sigma i)}$ on U_q. If σ is an order-preserving permutation of $\mathbf{Q}$, we define the action of σ on U by $\sigma u_{qi} = u_{(\sigma q)i}$, and observe that σ preserves $\langle U', < \rangle$ and all the second subscripts.

Let $U(q, G) = \{\sigma_q \mathrm{U}_q : \sigma \in G\}$ and $U_G = \{U(q, G) : q \in \mathbf{Q}\}$. Then in fact the permutation on U induced by any order-preserving permutation of $\mathbf{Q}$ preserves U_G.

As before, F is a generic choice function for $\{w(\xi) : \xi \in T_{Z_1}(U), |\xi| = n(w)\}$, and $\mathfrak{N}$ is the submodel of $\mathfrak{M}[F]$ consisting of all members of $\mathfrak{M}[F]$ hereditarily ordinal definable over $\{F, \langle U', < \rangle, U_G\} \cup U$.

This time Φ is the class of formulae involving ordinals, F, $\langle U', < \rangle$, and U_G as parameters. Φ can be well-ordered in $\mathfrak{N}$. The following lemma uses a combination of the methods of Lemma 3.4 and a lemma of Mostowski (see [4, p. 239]).

Lemma 3.10. *Let $X \in \mathfrak{N}$, and suppose that X is $\{F, \langle U', < \rangle, U_G\}$-ordinal definable over each of A and B, where A and B are finite unions of members of U'. Then X is $\{F, \langle U', < \rangle, U_G\}$-ordinal definable over $A \cap B$*

Proof. Let $A' = \{q : U_q \subseteq A\}$, $B' = \{q : U_q \subseteq B\}$, and suppose that $A' \not\subseteq B'$. Let $q \in A' - B'$. We show that X is $\{F, \langle U', < \rangle, U_G\}$-ordinal definable over $A - U_q$, and then use induction as before.

Let $X = \{x : \varphi(x, \mathbf{U}_q, u_1, ..., u_n)\} = \{x : \psi(x, \mathbf{B})\}$, where $\varphi, \psi \in \Phi$, $\{u_1, ..., u_n\} = A - U_q$ and $\mathbf{B}$ is an arrangement of B in a sequence.

Let $\theta(x_0, ..., x_{N-1}, x'_1, ..., x'_n)$ be a formula such that $\mathsf{p}_A \subseteq F$ is equivalent to $\theta(\mathbf{U}_q, u_1, ..., u_n)$, where $\theta \in \Phi$, p_A is the set of all members of p involving just members of A, and p is some condition in $\mathfrak{J}$ such that

$$(5) \qquad \mathsf{p} \Vdash (\forall x)(x \in X \leftrightarrow \varphi(x, \mathbf{U}_q, u_1, ..., u_n) \leftrightarrow \psi(x, \mathbf{B}))\,.$$

We let $\chi(x, x_1, ..., x_n)$ be the following formula:

$$(\exists y \in U')(\exists z_0, ..., z_{N-1})(\forall y_1, ..., y_n \in U')[y = \{z_0, ..., z_{N-1}\}$$
$$\wedge \langle z_0, ..., z_{N-1}\rangle \in \mathbf{U}U_G \wedge (x_1 \in y_1 \wedge ... \wedge x_n \in y_n \to$$
$$[yR_1y_1 \wedge yR_2y_2 \wedge ... \wedge yR_ny_n$$
$$\wedge\ \theta(z_0, ..., z_{N-1}, x_1, ..., x_n)$$
$$\wedge\ \varphi(x, z_0, ..., z_{N-1}, x_1, ..., x_n)])] \ ,$$

where each R_i is either $<$ or $>$, and is given as follows. Let $u_i \in U_{q_i} \in U'$ (The q_i are then unique.) R_i is $<$ if $q < q_i$, and R_i is $>$ if $q > q_i$.

Again we prove that $X = \{x\colon \chi(x, u_1, ..., u_n)\}$, and this will show that X is $\{F, \langle U', <\rangle, U_G\}$-ordinal definable over $A - U_q$ as desired.

If $x \in X$, then $\varphi(x, \mathbf{U}_q, u_1, ..., u_n)$. So we let $y = U_q$ and $z_i = u_{q_i}$. Since $\theta(\mathbf{U}_q, u_1, ..., u_n)$, we have $\theta(z_0, ..., z_{N-1}, u_1, ..., u_n)$. Also we must have $y_i = U_{q_i}$, and so yR_iy_i for each i, as desired. Finally, $\langle z_0, ..., z_{N-1}\rangle = \mathbf{U}_q \in \mathbf{U}U_G$. Therefore $X \subseteq \{x\colon \chi(x, u_1, ..., u_n)\}$.

Secondly, we suppose that $\chi(x, u_1, ..., u_n)$ and $x \notin X$. Then $\neg\varphi(x, \mathbf{U}_q, u_1, ..., u_n)$ holds. By definition of χ, there is a $y \in U'$, and $z_0, ..., z_{N-1} \in y$ such that $y = \{z_0, ..., z_{N-1}\}, \langle z_0, ..., z_{N-1}\rangle \in \mathbf{U}U_G$, and

$$\theta(z_0, ..., z_{N-1}, u_1, ..., u_n) \wedge \varphi(x, z_0, ..., z_{N-1}, u_1, ..., u_n).$$

Also $yR_1y_1 \wedge ... \wedge yR_ny_n$ shows that if $y = U_r$ and $u_i \in U_{q_i}$ for each i, then r is in the same interval of $\mathbf{Q}$ as q determined by the q_i. In particular, $r \notin A' - \{q\}$.

Let σ be an order-preserving permutation of $\mathbf{Q}$ which fixes A' pointwise, and such that r is in the same interval of $(A' - \{q\}) \cup \sigma''B'$ as q. We may also suppose that $B' \cap \sigma''B' \subseteq A'$.

Let τ be an order-preserving permutation of $\mathbf{Q}$ which fixes $(A' - \{q\}) \cup \sigma''B'$ pointwise, and maps q to r. The action of σ and τ is then induced on U, and preserves $\langle U', <\rangle$ and U_G. Using (5) and the appropriate analogue of Lemma 3.3, we have

(6) $\quad \sigma \mathrm{p}_{A\cup B} \Vdash (\forall x)(\varphi(x, \mathbf{U}_q, u_1, ..., u_n) \leftrightarrow \psi(x, \sigma\mathbf{B}))$,

(7) $\quad \tau\sigma \mathrm{p}_{A\cup B} \Vdash (\forall x)(\varphi(x, \mathbf{U}_r, u_1, ..., u_n) \leftrightarrow \psi(x, \sigma\mathbf{B})$.

Now as $\langle z_0, ..., z_{N-1}\rangle \in \mathbf{U}U_G$, there is a $\rho \in G$ such that

$$\langle z_0, ..., z_{N-1}\rangle = \rho_r \mathbf{U}_r .$$

Thus we have

$$(8) \qquad \rho_r \tau \sigma \mathsf{p}_{A\cup B} \Vdash (\forall x)(\varphi(x, z_0, ..., z_{N-1}, u_1, ..., u_n) \leftrightarrow \psi(x, \sigma\mathbf{B})) .$$

Since $\neg\varphi(x, \mathbf{U}_q, u_1, ..., u_n)$ and $\varphi(x, z_0, ..., z_{N-1}, u_1, ..., u_n)$ hold in $\mathfrak{N}$, there is a condition q in $\mathfrak{F}$ such that

$$\mathsf{q} \Vdash \neg(\forall x)(\varphi(x, \mathbf{U}_q, u_1, ..., u_n) \leftrightarrow \varphi(x, z_0, ..., z_{N-1}, u_1, ..., u_n)),$$

and by the analogue of Lemma 3.3,

$$(9) \qquad \mathsf{q}_{A\cup U_r} \Vdash \neg(\forall x)(\varphi(x, \mathbf{U}_q, u_1, ..., u_n) \leftrightarrow \varphi(x, z_0, ..., z_{N-1}, u_1, ..., u_n)) .$$

The proof that $\sigma\mathsf{p}_{A\cup B}$, $\rho_r\tau\sigma\mathsf{p}_{A\cup B}$, and $\mathsf{q}_{A\cup U_r}$ are compatible is precisely as in the proof of Theorem 3.2. Thus by (6), (8) and (9), $\sigma\mathsf{p}_{A\cup B} \cup \rho_r\tau\sigma\mathsf{p}_{A\cup B} \cup \mathsf{q}_{A\cup U_r}$ forces a contradiction.

This completes the proof of Lemma 3.10.

Thus for each $X \in \mathfrak{N}$, there is a unique minimal finite $A \subseteq U$ such that A is a union of members of U', and X is $\{F, \langle U', <\rangle, U_G\}$-ordinal definable over A. Let this A be $s(X)$, and φ_X, $s'(X)$, Ω be as before.

One shows that C_{Z_1} holds in $\mathfrak{N}$ exactly as in the proof of Theorem 3.2. Next we show that $C^0_{Z_2}$ holds in $\mathfrak{N}$.

Let $\langle X, <\rangle$ be an ordered set in $\mathfrak{N}$. Let A be the support of $\langle X, <\rangle$. It is easily seen that if a permutation σ of U fixes $\langle U', <\rangle$, U_G and every member of the support of a set Y, then σ fixes Y too.

Thus if σ is a permutation of U fixing $\langle U', <\rangle$, U_G and A pointwise, it fixes $\langle X, <\rangle$ too. Let us suppose in addition that σ has finite order, m say.

If $\xi \in X$ and $\sigma\xi \neq \xi$, then as $\sigma X = X$, $\sigma\xi \in X$, so $\xi < \sigma\xi$ or $\xi > \sigma\xi$. Since σ fixes $<$,

$$\xi < \sigma\xi < \sigma^2\xi < \ldots < \sigma^m\xi = \xi$$

or

$$\xi > \sigma\xi > \sigma^2\xi > \ldots > \sigma^m\xi = \xi\,,$$

each of them impossible. Hence σ fixes X pointwise.

Let $B = s(\xi) - A$, where $\xi \in X$, and $|\xi| = n(w)$ for some $w \in Z_2$. Let $|B| = N \cdot m$ and let F_ξ be the set of all $1-1$ mappings f from B onto $N \cdot m$ which take the j^{th} member of $B' = \{U_q \in U' \colon U_q \subseteq B\}$ in the ordering of U' $1-1$ onto the j^{th} copy of N, and moreover such that for every component f_j of this form, f_j maps some member of $U(q_j, G)$ to the j^{th} copy of $\langle 0, 1, ..., N-1\rangle$ (where U_{q_j} is the j^{th} member of B'). F_ξ is then in $\mathfrak{N}$, and if $f, f' \in F_\xi$, then $f'f^{-1} \in G^m$. (This is the reason for the somewhat elaborate definition of F_ξ.) G^m acts on $N \cdot m$ of course, and for each $f \in F_\xi$, we get the induced action on B. By the discussion above, any such induced permutation fixes ξ. Hence if $f \in F_\xi$, then $H(f(\xi)) = G^m$. Moreover, $f(\xi)$ is independent of the choice of f from F_ξ. By the choice of G, there is an $\eta \in w(f(\xi))$ such that $H(\eta) \supseteq G^m$.

Choose the least such η, η_ξ say, in $e_\omega(N \cdot m)$. Let $g(w(\xi)) = f^{-1}(\eta_\xi)$ for $f \in F_\xi$.

If f and f' are in F_ξ, then $f'f^{-1} \in G^m$ as we saw above, and so $f'f^{-1}\eta_\xi = \eta_\xi$. Thus $g(w(\xi))$ is independent of the choice of f from F_ξ.

But clearly $g(w(\xi)) \in f^{-1}(w(f(\xi))) = w(\xi)$. Hence g is the desired choice function, and $C^0_{Z_2}$ holds in $\mathfrak{N}$.

To show that $C^*_{Z_3}$ holds it is clearly enough to show that any well-ordered union of well-orderable sets can be well-ordered. The proof of this is the same as in Theorem 3.2 (see Lemma 3.8).

Finally, we show that C^0_v fails. $\langle U', <\rangle$ is in $\mathfrak{N}$, and it is an ordered set of $n(v)$-element sets. Suppose that $f \in \mathfrak{N}$ is a choice function for $\{v(\xi) \colon \xi \in U'\}$, and let A be the set of members of U occurring in p or $s(f)$, where p is a condition forcing "f is a choice function for $\{v(\xi) \colon \xi \in U'\}$".

Let $B \in U'$, $B \cap A = \emptyset$, and let q_1 be a choice function for $\{w(\xi) \colon \xi \in T_{Z_1}(A),\ |\xi| = n(w)\}$ extending p. By our assumption on G, its action G_B on B satisfies the following. G_B moves every member of $v(B)$, and for every $X \in e_\omega(B)$ with $|X| = n(w)$ for some $w \in Z_1$, there is a $\xi \in w(X)$ such that $H(X) \cap G \subseteq H(\xi)$. By Lemma 3.1, therefore there is an extension q of q_1 to $\{w(\xi): \xi \in T_{Z_1}(A \cup B), |\xi| = n(w)\}$ such that for each $x \in v(B)$, there is a permutation of $A \cup B$ fixing A pointwise, q, and U_G (since it lies in G), and moving x. This yields a contradiction as before.

In fact, as in Theorem 3.2 this argument shows that no infinite subset of $\{v(\xi): \xi \in U'\}$ has a choice function.

Theorem 3.11. $\mathrm{C}(Z_1, Z_2, v)$ *is necessary for the implication* $\mathrm{C}_{Z_1} \wedge \mathrm{C}^0_{Z_2} \to \mathrm{C}^*_v$.

Proof. Suppose that $\mathrm{C}(Z_1, Z_2, v)$ fails. Then there is a group G of permutations of $N = n(v)$ which moves every member of v, and such that whenever $X \in e_\omega(N \cdot m)$ for some $m \in \omega$, if $|X| = n(w)$, where $w \in Z_1$, then there is a $\xi \in w(X)$ such that $G^m \cap H(X) \subseteq H(\xi)$, and if $|X| = n(w)$, where $w \in Z_2$ and $H(X) = G^m$, then there is a $\xi \in w(X)$ such that $G^m \subseteq H(\xi)$.

This time we shall not use forcing, so there is no need to make the assumption that $\mathfrak{M}$ is countable. We do wish U to be countable, however, and we index it by $\omega \times N$. Thus

$$U = \{u_{ij}: i \in \omega, j \in N\}.$$

For each i, $U_i = \{u_{ij}: j \in N\}$, and $U' = \{U_i: i \in \omega\}$. U' receives the well-ordering $<$ induced by that of ω.

$\mathbf{U}_i$ is $\langle u_{i0}, ..., u_{iN-1}\rangle$, and if $\sigma \in G$, σ_i is the permutation of U which acts on U_i like σ, and is the identity elsewhere. $U(i, G) = \{\sigma_i \mathbf{U}_i: \sigma \in G\}$, and $U_G = \{U(i, G): i \in \omega\}$.

$\mathfrak{N}$ is the submodel of $\mathfrak{M}$ consisting of all members of $\mathfrak{M}$ hereditarily ordinal definable over $\{\langle U', <\rangle, U_G\} \cup U$. Φ is the class of all formulae involving $\langle U', <\rangle$, U_G and ordinals as parameters.

For each $X \in \mathfrak{N}$, $M(X)$ is the least $M \in \omega$ such that X is $\{\langle U', <\rangle, U_G\}$ ordinal definable over $\bigcup\{U_i: i < M\}$.

Now let X be any set of $\mathfrak{N}$. We must show that $\{w(\xi): \xi \in X, |\xi| = n(w), w \in Z_1\}$ has a choice function in $\mathfrak{N}$. Let $M(X) = M$. Then any permutation of U fixing $\langle U', <\rangle$, U_G and $\bigcup\{U_i: i < M\}$ pointwise fixes X too. Let $\xi \in X$, $|\xi| = n(w)$, where $w \in Z_1$, and let $m = M(\langle \xi, X\rangle) - M$. Thus each permutation of U which fixes $\bigcup\{U_i: i < M, i \geq M(\langle \xi, X\rangle)\}$ pointwise, and acts on each U_i like a member of G, fixes X.

Let us suppose for the moment that $X \subseteq T_{Z_1}(U)$. Let F_ξ be the set of all 1–1 mappings f from $B = \bigcup\{U_i: M \leq i < M(\langle\xi, X\rangle)\}$ onto $N \cdot m$ which take U_{M+j} 1–1 onto the j^{th} copy of N, and such that each $f_j = f \upharpoonright U_{M+j}$ maps some member of $U(M+j, G)$ to the j^{th} copy of $\langle 0, 1, ..., N-1\rangle$. $F_\xi \in \mathfrak{N}$ of course, and if $f, f' \in F_\xi$, then $f'f^{-1} \in G^m$.

G^m acts on $N \cdot m$ and for each $f \in F_\xi$, we get the induced action of G^m on $\bigcup\{U_i: M \leq i < M(\langle\xi, X\rangle)\}$. By the above remarks, each such induced permutation fixes X. Let $\overline{F}_\xi$ be the set of all f in F_ξ such that $f(\xi)$ is minimal. Thus for $f \in \overline{F}_\xi$, $f(\xi)$ is constant. By the choice of G, there is a least $\eta = \eta_\xi \in w(f(\xi))$ such that $G^m \cap H(f(\xi)) \subseteq H(\eta)$.

Let $g(w(\xi)) = f^{-1}\eta_\xi$ for $f \in \overline{F}_\xi$. Then $g(w(\xi)) \in w(\xi)$, and we just have to check that $g(w(\xi))$ is independent of the choice of f from $\overline{F}_\xi$.

If $f, f' \in \overline{F}_\xi$, then $f'f^{-1} \in G^m \cap H(f(\xi))$. By the choice of η_ξ, $G^m \cap H(f(\xi)) \subseteq H(\eta_\xi)$, and so $f'f^{-1}\eta_\xi = \eta_\xi$ as desired.

Thus whenever $X \subseteq T_{Z_1}(U)$, $\{w(\xi): \xi \in X, w \in Z_1, |\xi| = n(w)\}$ has a choice function. But this induces a similar choice function corresponding to an arbitrary X by the argument used in Theorem 3.2. Hence C_{Z_1} holds in $\mathfrak{N}$.

The proof that $C^0_{Z_2}$ holds in $\mathfrak{N}$ is just as for the similar proof in Theorem 3.9.

It remains to be shown that C^*_v fails in $\mathfrak{N}$. Suppose that f is a choice function for $\{v(\xi): \xi \in U'\}$. Then if $M = M(f)$, any permutation of U which fixes $\langle U', <\rangle$, U_G and $\bigcup\{u_i: i < M\}$ pointwise also fixes f. But this is contrary to the choice of G. As before, this shows in fact that no infinite subset of $\{v(\xi): \xi \in U'\}$ has a choice function.

§4. Effectiveness of the conditions and some special cases

We firstly summarize the results of the theorems of Section 2 and Section 3 in the following way.

Theorem 4.1. (i) $C_{Z_1} \wedge C^0_{Z_2} \wedge C^*_{Z_3} \to C_v$ *if and only if* $\mathsf{A}(Z_1, v)$.
(ii) $C_{Z_1} \wedge C^0_{Z_2} \wedge C^*_{Z_3} \to C^0_v$ *if and only if* $\mathsf{B}(Z_1, Z_2, v)$.
(iii) $C_{Z_1} \wedge C^0_{Z_2} \wedge C^*_{Z_3} \to C^*_v$ *if and only if* $\mathsf{C}(Z_1, Z_2 \cup Z_3, v)$.

Proof. (i) If $C_{Z_1} \wedge C^0_{Z_2} \wedge C^*_{Z_3} \to C_v$, then by Theorem 3.2, and the obvious fact that $C^0_{Z_3} \to C^*_{Z_3}$, we have $C_{Z_1} \wedge C^0_{Z_2 \cup Z_3} \to C_v$ and so $\mathsf{A}(Z_1, v)$ must hold.

Conversely, if $\mathsf{A}(Z_1, v)$ holds, $C_{Z_1} \to C_v$ by Theorem 2.1, and so $C_{Z_1} \wedge C^0_{Z_2} \wedge C^*_{Z_3} \to C_v$.

(ii) follows from Theorems 2.3 and 3.9 in a similar way, and (iii) from Theorems 2.4 and 3.11.

Lemma 4.2. $\mathsf{A}(Z, v)$ *holds if and only if there is a finite* $Z' \subseteq Z$ *such that* $\mathsf{A}(Z', v)$*. Similarly for* $\mathsf{B}(Z_1, Z_2, v)$ *and* $\mathsf{C}(Z_1, Z_2, v)$.

Proof. If there is a finite $Z' \subseteq Z$ such that $\mathsf{A}(Z', v)$ holds, then clearly $\mathsf{A}(Z, v)$ holds. Conversely, suppose that $\mathsf{A}(Z, v)$ holds. For each group G of permutations of $n(v)$ which moves every member of v, choose a $w \in Z$ such that for some $X \in e_\omega(n(v))$ with $|X| = n(w)$, we have $H(X) \cap G \nsubseteq H(\xi)$ for each $\xi \in w(X)$. Let Z' be the set of all w chosen in this way. As $S(n(v))$ has only finitely many subgroups, Z' is finite. Clearly $\mathsf{A}(Z', v)$ holds.

Similarly for $\mathsf{B}(Z_1, Z_2, v)$ and $\mathsf{C}(Z_1, Z_2, v)$.

Theorem 4.3. (i) $C_{Z_1} \wedge C^0_{Z_2} \wedge C^*_{Z_3} \to C_v$ *if and only if for some finite* $Z \subseteq Z_1$, $C_Z \to C_v$.
(ii) $C_{Z_1} \wedge C^0_{Z_2} \wedge C^*_{Z_3} \to C^0_v$ *if and only if for some finite* $Z'_1 \subseteq Z_1$ *and* $Z'_2 \subseteq Z_2$, $C_{Z'_1} \wedge C^0_{Z'_2} \to C^0_v$.
(iii) $C_{Z_1} \wedge C^0_{Z_2} \wedge C^*_{Z_3} \to C^*_v$ *if and only if for some finite* $Z'_1 \subseteq Z_1$ *and* $Z'_2 \subseteq Z_2 \cup Z_3$, $C_{Z'_1} \wedge C^*_{Z'_2} \to C^*_v$.

Lemma 4.4. *If* Z, Z_1, Z_2 *are finite, then* $\mathsf{A}(Z, v)$, $\mathsf{B}(Z_1, Z_2, v)$ *and* $\mathsf{C}(Z_1, Z_2, v)$ *are effectively verifiable.*

Proof. We use Lemma 2.2. Firstly consider $\mathsf{A}(Z, v)$. We may restrict our search for an appropriate X to $e_{k+5}(n(v))$, where $k = \max\{n(w): w \in Z\}$, and so there is a finite procedure for determining the validity of $\mathsf{A}(Z, v)$.

A similar method suffices for $\mathsf{B}(Z_1, Z_2, v)$ with the additional remark that we may also restrict our search to $e_\omega(n(v) \cdot m)$ for m a fixed positive integer (depending on Z_2). We omit a detailed proof of this but indicate what is involved. Given an arbitrary $X \in e_\omega(n(v) \cdot m)$, where $|X| = n(w)$ and $w \in Z_2$, such that $H(X) = G^m$ and for each $\xi \in w(X)$, $G^m \subseteq H(\xi)$, one constructs such an X in $e_\omega(n(v) \cdot N)$, where $N = |w(X)|$. (N is independent of X of course.) For $G^m \not\subseteq H(\xi)$ if and only if for some coordinate j, $G_j \not\subseteq H(\xi)$, and so for each $\xi \in w(X)$, one needs just one coordinate to satisfy $G_j \not\subseteq H(\xi)$, and hence at most $|w(X)|$ in all.

The same method applies to $\mathsf{C}(Z_1, Z_2, v)$.

Corollary 4.5. *There is an effective procedure for determining the validity of each of* $\mathrm{C}_{Z_1} \wedge \mathrm{C}^0_{Z_2} \wedge \mathrm{C}^*_{Z_3} \to \mathrm{C}_v, \mathrm{C}^0_v, \mathrm{C}^*_v$ *when the* Z_i *are finite.*

As we remarked earlier, the C_n are a particular case of the C_v, and so we are able to deduce various results about the C_n. The following strengthens Gauntt's result of [1] that the axiom of choice for well-ordered sets of finite sets ($\mathrm{C}^*_{\omega-\{0\}}$) does not imply any C_n for $n > 1$.

Corollary 4.6. (i) *The axiom of choice for ordered sets of finite sets* $\mathrm{C}^0_{\omega-\{0\}}$ *does not imply any* C_n *with* $n > 1$.

(ii) *The axiom of choice for well-ordered sets of finite sets* $\mathrm{C}^*_{\omega-\{0\}}$ *does not imply any* C^0_n *with* $n > 1$.

Proof. (i) By Theorem 4.1(i), $\mathrm{C}^0_{\omega-\{0\}} \to \mathrm{C}_n$ if and only if $\mathsf{A}(\emptyset, n)$. But clearly if $n > 1$, $\mathsf{A}(\emptyset, n)$ fails.

(ii) By Theorem 4.1(ii), $\mathrm{C}^*_{\omega-\{0\}} \to \mathrm{C}^0_n$ if and only if $\mathsf{B}(\emptyset, \emptyset, n)$, and again this fails when $n > 1$.

Corollary 4.7. (i) $\mathrm{C}^0_Z \to \mathrm{C}^*_v \Leftrightarrow \mathrm{C}^*_v \to \mathrm{C}^*_v \Leftrightarrow \mathrm{C}^0_Z \to \mathrm{C}^0_v$.

(ii) $\mathrm{C}_Z \to \mathrm{C}^0_v \Leftrightarrow \mathrm{C}_Z \to \mathrm{C}_v$.

(*These are two of the results mentioned in the introduction.*)

Proof. (i) By Theorem 4.1(ii) and (iii),

$C_Z^0 \to C_v^* \Leftrightarrow \mathsf{C}(\emptyset, Z, v)$,

$C_Z^0 \to C_v^* \Leftrightarrow \mathsf{C}(\emptyset, Z, v)$,

$C_Z^0 \to C_v^0 \Leftrightarrow \mathsf{B}(\emptyset, Z, v)$.

But $\mathsf{B}(\emptyset, Z, v) \leftrightarrow \mathsf{C}(\emptyset, Z, v)$, so all three are equivalent.

(ii) $C_Z \to C_v^0 \Leftrightarrow \mathsf{B}(Z, \emptyset, v)$,

$C_Z \to C_v \Leftrightarrow \mathsf{A}(Z, v)$.

We just observe therefore that $\mathsf{A}(Z, v) \Leftrightarrow \mathsf{B}(Z, \emptyset, v)$.

Various other similar interconnections follow, for example, Gauntt's result of [1] that $C_Z^* \to C_v^*$ implies $C_Z \to C_v$, but we do not bother to list them all.

We conjecture that if whenever $\xi \in e_\omega(X)$, $|\xi| = n(w)$ for some $w \in Z$, and $H(\xi) = G$, there is an $\eta \in w(\xi)$ such that $G \subseteq H(\eta)$, then also whenever $\xi \in e_\omega(X \cdot m)$ for some $m \in \omega$, $|\xi| = n(w)$ for some $w \in Z$, and $H(\xi) = G^m$, then there is an $\eta \in w(\xi)$ such that $G^m \subseteq H(\eta)$.

We have been able to prove this only in the case where $w(\xi) = \xi$ for each $w \in Z$ (i.e., in the original number-theoretic setting).

Lemma 4.8. *Suppose that $Z \subseteq \omega$, $n \in \omega$, and that whenever $\xi \in e_\omega(n)$, $|\xi| \in Z$, and $H(\xi) = G$, a group of permutations of n, there is an $\eta \in \xi$ such that $G \subseteq H(\eta)$. Then also whenever $\xi \in e_\omega(n \cdot m)$ for any $m \in \omega$, $|\xi| \in Z$, and $H(\xi) = G^m$, there is an $\eta \in \xi$ such that $G^m \subseteq H(\eta)$.*

Proof. Suppose the contrary, and let $\xi \in e_\omega(n \cdot m)$, $|\xi| \in Z$ be such that $H(\xi) = G^m$ and for every $\eta \in \xi$, $G^m \not\subseteq H(\eta)$.

Let $\eta_1, \ldots, \eta_k$ be representatives of the G^m-orbits of ξ. For each i, $G^m \not\subseteq H(\eta_i)$. So for some least copy j_i of G, $G_{j_i} \not\subseteq H(\eta_i)$. One constructs in the manner of Lemma 2.2 an element $\zeta_i \in e_\omega(n)$ such that $H(\zeta_i) = G_{j_i} \cap H(\eta_i)$. Also it is clear that $|G_{j_i} : G_{j_i} \cap H(\eta_i)|$ divides $|G^m : G^m \cap H(\eta_i)|$, and so we let

$$n_i \cdot |G_{j_i} : G_{j_i} \cap H(\eta_i)| = |G^m : G^m \cap H(\eta_i)|\,.$$

Now let $Y = \{\langle \sigma\zeta_i, i, j\rangle : j < n_i,\ 1 \le i \le k,\ \sigma \in G\}$. Then

$$|Y| = \sum_{i=1}^{k} |G : H(\zeta_i)| \cdot n_i = \sum_{i=1}^{k} |G^m : G^m \cap H(\eta_i)| = |\xi| \in Z\,.$$

But $H(Y) = G$, and for every $\zeta \in Y$, $G \not\subseteq H(\zeta)$, which is a contradiction.

Lemma 4.9. *If $Z \subseteq \omega$, then:*
(i) $\mathsf{D}(n, Z) \Leftrightarrow \mathsf{A}(Z, n)$,
(ii) $\mathsf{L}(n, Z) \Leftrightarrow \mathsf{B}(\emptyset, Z, n) \Leftrightarrow \mathsf{C}(\emptyset, Z, n)$,
(iii) $\mathsf{K}(n, Z) \Leftrightarrow \mathsf{C}(Z, \emptyset, n)$.

Proof. We just prove (i). The other two parts are proved similarly. Lemma 4.8 is used in the proof of (ii).

Assume $\mathsf{D}(n, Z)$, and let G be a group of permutations of n which moves every member of n. By $\mathsf{D}(n, Z)$, there is a subgroup H of G, and proper subgroups $K_1, ..., K_r$ of H such that $\Sigma\,|H : K_i| \in Z$. For each i, $1 \leq i \leq r$, let $\xi_i \in e_\omega(n)$ satisfy $H(\xi_i) = K_i$.

Then we let $X = \{\langle \sigma\xi_i, i\rangle : \sigma \in H,\ 1 \leq i \leq r\}$. We have

$$|X| = \sum_{i=1}^{r} |H : K_i| \in Z\ ,$$

and $H(Z) \supseteq H$.

If $\xi \in X$, $\xi = \langle \sigma\xi_i, i\rangle$ say, then $H(\xi) = H(\sigma\xi_i) = \sigma H(\xi_i)\sigma^{-1}$, a proper subgroup of H. Thus X is as desired for $\mathsf{A}(Z, n)$.

Conversely, assume $\mathsf{A}(Z, n)$, and let G be a subgroup of S_n without fixed points. By $\mathsf{A}(Z, n)$, there is an $X \in e_\omega(n)$ such that $|X| \in Z$, and for each $\xi \in X$, $H(X) \cap G \not\subseteq H(\xi)$. Let $\xi_1, ..., \xi_r$ be representatives of the $H(X) \cap G$-orbits of X. Let $H = H(X) \cap G$ and $K_i = H(X) \cap G \cap H(\xi_i)$ Then each K_i is a proper subgroup of H, and

$$|X| = \sum_{i=1}^{r} |H : K_i|\ .$$

Thus $\mathsf{D}(n, Z)$ holds.

Corollary 4.10.
(i) $\mathsf{D}(n, Z) \Leftrightarrow \mathsf{C}_Z \to \mathsf{C}_n$ *(Mostowski–Gauntt).*
(ii) $\mathsf{L}(n, Z) \Leftrightarrow \mathsf{C}^*_Z \to \mathsf{C}^*_n$ *(Gauntt).*
(iii) $\mathsf{K}(n, Z) \Leftrightarrow \mathsf{C}_Z \to \mathsf{C}^*_n$.

(iv) $\mathsf{L}(n, Z) \Leftrightarrow C_Z^0 \to C_n^0 \Leftrightarrow C_Z^0 \to C_n^*$.
(v) $\mathsf{D}(n, Z) \Leftrightarrow C_Z \to C_n^0$.

One would like to have conditions which were easier to handle than D, L and K for dealing with numerical cases, and one very suggestive one was proposed by Mostowski in [5].

$\mathsf{M}(n, Z)$: If $n = p_1 + \ldots + p_s$ is any expression for n as a sum of (not necessarily distinct) primes, then there are $\alpha_i \in \omega$ such that $\Sigma \alpha_i p_i \in Z$.

Now it is easy to see that $\mathsf{K}(n, Z) \to \mathsf{M}(n, Z)$, and so $\mathsf{M}(n, Z)$ is certainly necessary for $C_Z \to C_n^*$. The converse, $\mathsf{M}(n, Z) \to \mathsf{K}(n, Z)$ has been proved by Dr. M.J. Collins. This gives of course the sufficiency of $\mathsf{M}(n, Z)$ for $C_Z \to C_n^*$. We sketch his proof here. (Professor Mostowski informed us that this was also proved by Dr. K. Wiśniewski in [10].)

Assume $\mathsf{M}(n, Z)$ and let G be a fixed point free group of permutations of n. We must show that for some $m \in \omega$, $H \subseteq G^m$ and proper subgroups $K_1, \ldots, K_r$ of H, $\Sigma |H : K_i| \in Z$. The first remark is that it is enough to find a collection of pairs $(H_1, K_1), \ldots, (H_r, K_r)$ such that $H_i \subseteq G$ for each i, and each K_i is a proper subgroup of H_i of prime index, p_i say, where $\Sigma p_i = n$. For then by $\mathsf{M}(n, Z)$, there are $\alpha_i \in \omega$ such that $\Sigma \alpha_i p_i \in Z$. Let $H = H_1 \times \ldots \times H_r$, and $K_i' = H_1 \times \ldots \times H_{i-1} \times K_i \times H_{i+1} \times \ldots \times H_r$. Then taking each $K_i' \alpha_i$ times we obtain the desired result.

Now let $\eta_1, \ldots, \eta_m$ be the G-orbits of n and let $|\eta_i| = n_i$. Since G is fixed point free, $n_i > 1$ and we may let p_i be a prime factor of n_i. Let N_i be the kernel of the action of G on η_i, that is, the subgroup of G fixing η_i pointwise. Then $n_i \mid |G : N_i|$ since G acts transitively on η_i, and so $p_i \mid |G : N_i|$. Since N_i is normal in G, there is a subgroup M_i of G containing N_i such that $|M_i : N_i| = p_i$. Take each such pair (M_i, N_i) n_i/p_i times for $i = 1, \ldots, m$.

Corollary 4.11. $\mathsf{M}(n, Z) \Leftrightarrow C_Z \to C_n^*$.

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